10. P. Chen, Detached Flows [Russian translation], Vol. 2, Mir, Moscow (1973).
11. W. D. Hayes and R. F. Probstein, Hypersonic Flow Theory [Russian translation], Inostr. Lit., Moscow (1962).
12. J. A. Fay and F. R. Riddell, "Theory of stagnation point heat transfer in dissociated air," J. Aeronaut. Sci., 25, No. 2 (1958).
13. Yu. V. Polezhaev, V. P. Romanchikov, I. V. Chirkov, and V. N. Shebeko, "Calculation model for erosion destruction of composite material," Inzh. Fiz. Zh., 37, No. 3 (1979).
14. J. K. Boyson and H. A. Curtiss, "Experimental studies of the velocity gradient on a blunt body," in: Mechanics [Russian translation], Inostr. Lit., Moscow (1960).
15. K. M. Magomedov, "Supersonic flow over blunt bodies with a known sonic point," Izv. Akad. Nauk SSSR, OTN, Mekh. Mashinstr., No. 1 (1963).
16. R. E. Phinney, "Mechanism for heat transfer to a rough blunt body," Lett. Heat Mass Transf., 1, No. 2 (1974).

GENERAL SOLUTIONS AND REDUCTION OF A SYSTEM OF EQUATIONS OF THE LINEAR THEORY OF ELASTICITY TO DIAGONAL FORM

N. I. Ostrosablin

UDC 539.3,517.958

Numerous attempts have been made [1-11] to represent stresses or displacements in terms of arbitrary independent functions (for example, harmonic and biharmonic functions) in such a way that the equations of elasticity theory are satisfied identically. We call such representations general solutions. However, up to the present, there has been no single approach to the construction of general solutions. In the present paper we present a method which makes it possible to reduce, in certain cases, a system of differential equations (of linear elasticity theory) with constant coefficients to a simpler system; in particular, to a diagonal system. Moreover, the transformation inverse to the initial system is specified by a transposed or conjugate matrix. Expressions are also obtained for the production of new solutions (operators of symmetry in the sense of group analysis), starting from some concrete solution. The idea of the method is presented briefly in [12]. Explicit formulas are presented for isotropic and transversally isotropic materials, and completeness and generality of the Papkovich–Neiber solution is shown.

The equations of elasticity theory, in the presence of arbitrary anisotropy and the absence of volume forces, have the following form [7] in Cartesian orthogonal coordinates x_1, x_2, x_3:

$$L_{ij}u_j = 0, \quad L_{ij} = L_{ji} = A_{i(kl)j}\partial_{kl} - \rho\delta_{ij}\partial_{..}, \tag{1}$$

where u_j is the displacement vector; $A_{i(k\ell)j} = (A_{ik\ell j} + A_{i\ell kj})/2$; $A_{ik\ell j}$ is a constant tensor of elastic moduli; ρ is the constant density of the material; δ_{ij} is the Kronecker symbol; ∂_k indicates differentiation with respect to the coordinate x_k; and $\partial_{\dot{}}$ indicates differentiation with respect to the time; repeated subscripts indicate summation. Properties of the coefficients $A_{i(k\ell)j}$ were studied in [13-15].

We assume that the matrix L of the operators in relations (1) is similar [16] to some matrix D, i.e., a nondegenerate matrix T exists such that

$$LT = TD. \tag{2}$$

Since L' = L and we assume that D' = D, then from Eq. (2) we obtain

$$T'L = DT' \tag{3}$$

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 5, pp. 112-122, September-October, 1993. Original article submitted October 16, 1992.

 0021-8944/93/3405-0700$12.50

(D, T are matrices of operators with constant coefficients; primes denote transposition).

If u = Tv (v_j are new functions), where Dv = 0, then, taking Eq. (2) into account, we see that Eq. (1) is satisfied:

$$Lu = LTv = TDv = 0. \tag{4}$$

But if $v = T'\tilde{u}$, where $L\tilde{u} = 0$, then, taking Eq. (3) into account, we see that the following equation is satisfied:

$$Dv = DT'\tilde{u} = T'L\tilde{u} = 0. \tag{5}$$

Thus, in accordance with the formulas

$$u = Tv, \quad v = T'\tilde{u}, \tag{6}$$

solutions of Eqs. (4) and (5) pass over from one into the other and the systems (4) and (5) are equivalent [16].

We rewrite relation (2):

$$L_{ij}t_{jr} = t_{ip}D_{pr}. \tag{7}$$

Since L' = L, we can then assume that D is a diagonal matrix. Relation (7) then yields

$$L_{ij}t_{j1} = t_{i1}D_{11}, \quad L_{ij}t_{j2} = t_{i2}D_{22}, \quad L_{ij}t_{j3} = t_{i3}D_{33},$$

or

$$(L_{ij} - D\delta_{ij})\, t_j = 0, \tag{8}$$

i.e., we have obtained a problem for the characteristic operators $D_{11} = D_1$, ..., and vectors t_{j1}, ..., for the matrix of operators L_{ij}. A problem of this kind for operators of the theory of elasticity (1) was first posed in [9]. By virtue of symmetry of L_{ij}, we can assume that the characteristic vectors t_{j1}, t_{j2}, t_{j3} are orthogonal.

If the t_{jp} are numbers, the formulas (6) then correspond to an orthogonal transformation of coordinates and system (1) becomes a diagonal system for a special orthotropic material when $A_{2211} = -A_{1221}$, $A_{3311} = -A_{1331}$, $A_{3322} = -A_{2332}$. This case was presented in [15].

Of more interest is the version in which t_{jp} are operators. We consider first an isotropic material for which

$$L_{ij} = (\lambda + \mu)\,\partial_{ij} + (\mu\partial_{kk} - \rho\partial_{..})\,\delta_{ij} \tag{9}$$

(λ, μ are Lamé coefficients). It was shown in [9] that

$$D_1 = (\lambda + 2\mu)\,\partial_{kk} - \rho\partial_{..}, \quad D_2 = D_3 = \mu\partial_{kk} - \rho\partial_{..} \tag{10}$$

are characteristic operators for the matrix (9), and the characteristic vectors, to within arbitrary multipliers, are [9, 12]

$$t_{j1} = \partial_j, \quad t_{j2} = \varepsilon_{jps}c_p\partial_s, \quad t_{j3} = c_j\partial_{kk} - c_m\partial_{mj}, \tag{11}$$

where the ε_{jps} are Levi-Civita symbols; c_j is an arbitrary nonzero numerical vector or an operator vector with constant coefficients. The vectors (11) are orthogonal:

$$t_{ip}t_{iq} = \partial_{jj}\delta_{p1}\delta_{q1} + (c_jc_j\partial_{kk} - c_mc_n\partial_{mn})\,\delta_{p2}\delta_{q2} + \partial_{nn}t_{k2}t_{k2}\delta_{p3}\delta_{q3}, \tag{12}$$

where $|T| = t_{j3}t_{j3} = (t_{i1}t_{i1})(t_{j2}t_{j2}) = \partial_{ii}(c_jc_j\partial_{kk} - c_mc_n\partial_{mn})$ and

$$t_{in}t_{jn} = \partial_{ij} + \varepsilon_{ips}\varepsilon_{jqr}c_pc_q\partial_{sr} + (c_i\partial_{kk} - c_m\partial_{mi})\,(c_j\partial_{ss} - c_n\partial_{nj}). \tag{13}$$

Taking relations (4)-(6), (9)-(11) into account, we obtain, for an isotropic material, a solution of the Lamé equations (1), (9) in the following form:

$$u_i = \partial_i v_1 + \varepsilon_{ips}c_p\partial_s v_2 + (c_i\partial_{kk} - c_m\partial_{mi})\,v_3; \tag{14}$$

$$v_1 = \partial_j \tilde{u}_j, \quad v_2 = \varepsilon_{jps} c_p \partial_s \tilde{u}_j, \quad v_3 = (c_j \partial_{kk} - c_m \partial_{mj}) \tilde{u}_j; \tag{15}$$

$$[(\lambda + 2\mu) \partial_{kk} - \rho \partial_{..}] v_1 = 0, \tag{16}$$
$$(\mu \partial_{kk} - \rho \partial_{..}) v_2 = 0, \quad (\mu \partial_{kk} - \rho \partial_{..}) v_3 = 0;$$

$$[(\lambda + \mu) \partial_{ij} + (\mu \partial_{kk} - \rho \partial_{..}) \delta_{ij}] \tilde{u}_j = 0. \tag{17}$$

The Lamé system (1), (9), or (17) is equivalent [16] to the three independent equations (16), i.e., if v_j satisfies Eqs. (16), then the displacements u_i in Eq. (14) are a solution of the Lamé equations (1), (9) or (17), and, conversely, if $\tilde{u}_j$ is a solution of system (17), then the functions v_j in Eqs. (15) constitute a solution of the wave equations (16). In statics, obviously, the functions v_j are harmonic.

In the case of plane deformation $u_3 = 0$, $\partial_3 = 0$ and, for $c_1 = 0$, $c_2 = 0$, $c_3 = 1$, we obtain from relations (14), (15)

$$u_1 = \partial_1 v_1 - \partial_2 v_2, \quad u_2 = \partial_2 v_1 + \partial_1 v_2; \tag{18}$$

$$v_1 = \partial_1 \tilde{u}_1 + \partial_2 \tilde{u}_2, \quad v_2 = -\partial_2 \tilde{u}_1 + \partial_1 \tilde{u}_2. \tag{19}$$

We write formulas (18), (19) in the form of the complex combinations

$$u_1 + iu_2 = (\partial_1 + i\partial_2)(v_1 + iv_2) \equiv 2\partial_{\bar{z}}(v_1 + iv_2); \tag{20}$$
$$v_1 + iv_2 = (\partial_1 - i\partial_2)(\tilde{u}_1 + i\tilde{u}_2) \equiv 2\partial_z(\tilde{u}_1 + i\tilde{u}_2).$$

Here $i = \sqrt{-1}$; $z = x_1 + ix_2$. In statics v_1, v_2 are harmonic functions and we can take them in the form of the real part of analytic functions: $v_1 = \mathrm{Re}\,\varphi_1(z)$, $v_2 = \mathrm{Re}\,\varphi_2(z)$. From relations (20) we then have

$$u_1 + iu_2 = \overline{\varphi_1'(z)} + i\,\overline{\varphi_2'(z)}, \quad u_1 - iu_2 = \varphi_1'(z) - i\varphi_2'(z), \tag{21}$$

where the prime indicates differentiation with respect to z. The displacement representation (21) is a particular case of the Kolosov-Muskhelishvili formula [5].

We turn now to the general case of Eqs. (1). If relations (2) and (3) are satisfied and $L\tilde{u} = 0$, then $u = TT'\tilde{u}$ is also a solution:

$$Lu = LTT'\tilde{u} = TDT'\tilde{u} = TT'L\tilde{u} = 0.$$

If $D\tilde{v} = 0$, then $v = T'T\tilde{v}$ is also a solution:

$$Dv = DT'T\tilde{v} = T'LT\tilde{v} = T'TD\tilde{v} = 0.$$

The relation $u = TT'\tilde{u}$ is a formula for producing solutions, since for an arbitrary given solution $\tilde{u}$ we obtain a new solution u. This formula can be applied repeatedly. For an isotropic material the matrices T'T and TT' have the form (12), (13).

If $L\tilde{u} = 0$ and $LQ - QL = RL$ (Q is a symmetry operator [17]), then $u = Q\tilde{u}$ is also a solution: $Lu = LQ\tilde{u} = (Q + R)L\tilde{u} = 0$. It is evident from this that $Q = TT'$ is a symmetry operator on the group analysis sense, where $R = 0$.

The symmetry operator can also be taken in the form $Q = \ell M$, $Q + R = M\ell$ (M is an arbitrary matrix of operators with constant coefficients, $\ell = \ell'$ is the matrix of algebraic complements of elements L_{ij} in L). We then have $LQ = L\ell M = |L|M$, $(Q + R)L = M\ell L = M|L|$, i.e., the relation $LQ = (Q + R)L$ is satisfied.

For an isotropic material one of the matrices Q is the following [12]:

$$Q = \begin{bmatrix} q_{11} & [(\partial_{kk} - 2\partial_{33})\varphi - \psi]\partial_3 & [(\partial_{kk} - 2\partial_{22})\varphi + \psi]\partial_2 \\ [(\partial_{kk} - 2\partial_{33})\varphi + \psi]\partial_3 & q_{11} & [(\partial_{kk} - 2\partial_{11})\varphi - \psi]\partial_1 \\ [(\partial_{kk} - 2\partial_{22})\varphi - \psi]\partial_2 & [(\partial_{kk} - 2\partial_{11})\varphi + \psi]\partial_1 & q_{11} \end{bmatrix}.$$

Here q_{11}, φ, ψ are arbitrary operators with constant coefficients.

Equivalence of systems (4) and (5) does not mean a one-to-one correspondence between solutions of these systems; this would be the case when $|T|$ = const. In the general case solutions of systems (4) and (5) split up into non-intersecting classes of equivalent solutions between which a one-to-one correspondence is already established [16].

We turn now to Eq. (8). Let $D = a_{k\ell}\partial_{k\ell} - \rho\partial_{..}$, $a_{k\ell} = a_{\ell k}$, $t_j = \gamma_{js}\partial_s$. We may then write Eq. (8) in the form

$$(A_{i(kl)j} - \delta_{ij}a_{kl})\,\gamma_{js}\partial_{kls} = 0. \tag{22}$$

Collecting similar terms and setting the coefficients of $\partial_{k\ell s}$ to zero, from Eq. (22) we obtain

$$(A_{i(11)j} - \delta_{ij}a_{11})\,\gamma_{j1} = 0,\quad (A_{i(22)j} - \delta_{ij}a_{22})\,\gamma_{j2} = 0,\quad (A_{i(33)j} - \delta_{ij}a_{33})\,\gamma_{j3} = 0, \tag{23}$$

$$\begin{cases} 2\,(A_{i(23)j} - \delta_{ij}a_{23})\,\gamma_{j2} + (A_{i(22)j} - \delta_{ij}a_{22})\,\gamma_{j3} = 0,\\ (A_{i(33)j} - \delta_{ij}a_{33})\,\gamma_{j2} + 2\,(A_{i(23)j} - \delta_{ij}a_{23})\,\gamma_{j3} = 0,\end{cases}$$

$$\begin{cases} 2\,(A_{i(13)j} - \delta_{ij}a_{13})\,\gamma_{j1} + (A_{i(11)j} - \delta_{ij}a_{11})\,\gamma_{j3} = 0,\\ (A_{i(33)j} - \delta_{ij}a_{33})\,\gamma_{j1} + 2\,(A_{i(13)j} - \delta_{ij}a_{13})\,\gamma_{j3} = 0,\end{cases}$$

$$\begin{cases} 2\,(A_{i(12)j} - \delta_{ij}a_{12})\,\gamma_{j1} + (A_{i(11)j} - \delta_{ij}a_{11})\,\gamma_{j2} = 0,\\ (A_{i(22)j} - \delta_{ij}a_{22})\,\gamma_{j1} + 2\,(A_{i(12)j} - \delta_{ij}a_{12})\,\gamma_{j2} = 0,\end{cases}$$

$$2\,[(A_{i(23)j} - \delta_{ij}a_{23})\,\gamma_{j1} + (A_{i(13)j} - \delta_{ij}a_{13})\,\gamma_{j2} + (A_{i(12)j} - \delta_{ij}a_{12})\,\gamma_{j3}] = 0.$$

If we introduce the notation

$$A^{(1)} = A_{i(11)j},\quad A^{(2)} = A_{i(22)j},\quad A^{(3)} = A_{i(33)j},$$
$$A^{(4)} = \sqrt{2}A_{i(23)j},\quad A^{(5)} = \sqrt{2}A_{i(13)j},\quad A^{(6)} = \sqrt{2}A_{i(12)j},$$
$$a_1 = a_{11},\quad a_2 = a_{22},\quad a_3 = a_{33},\quad a_4 = \sqrt{2}a_{23},\quad a_5 = \sqrt{2}a_{13},$$
$$a_6 = \sqrt{2}a_{12},\quad \gamma_1 = \gamma_{j1},\quad \gamma_2 = \gamma_{j2},\quad \gamma_3 = \gamma_{j3},$$

system (23) may then be written in the matrix-block form (E is the unit matrix)

$$(A^{(1)} - Ea_1)\,\gamma_1 = 0,\quad (A^{(2)} - Ea_2)\,\gamma_2 = 0,\quad (A^{(3)} - Ea_3)\,\gamma_3 = 0, \tag{24}$$

$$\begin{bmatrix} \sqrt{2}\,(A^{(4)} - Ea_4) & A^{(2)} - Ea_2 \\ A^{(3)} - Ea_3 & \sqrt{2}\,(A^{(4)} - Ea_4)\end{bmatrix}\begin{bmatrix}\gamma_2\\ \gamma_3\end{bmatrix} = 0,$$

$$\begin{bmatrix} \sqrt{2}\,(A^{(5)} - Ea_5) & A^{(1)} - Ea_1 \\ A^{(3)} - Ea_3 & \sqrt{2}\,(A^{(5)} - Ea_5)\end{bmatrix}\begin{bmatrix}\gamma_1\\ \gamma_3\end{bmatrix} = 0,$$

$$\begin{bmatrix} \sqrt{2}\,(A^{(6)} - Ea_6) & A^{(1)} - Ea_1 \\ A^{(2)} - Ea_2 & \sqrt{2}\,(A^{(6)} - Ea_6)\end{bmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix} = 0,$$

$$\sqrt{2}\,[A^{(4)} - Ea_4\quad A^{(5)} - Ea_5\quad A^{(6)} - Ea_6]\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{bmatrix} = 0.$$

If $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 0$, i.e., if all these columns of matrix γ_{js} are equal to zero, Eqs. (24) are then satisfied. Then $t_j = 0$, i.e., the characteristic vector is the zero vector; however, this is inappropriate for us. Therefore, not all three columns can be simultaneously zero.

Assume now, for example, that two columns are equal to zero: $\gamma_2 = 0$, $\gamma_3 = 0$; then $t_j = \gamma_{j1}\partial_1$. Since t_j is determined to within a factor, the factor ∂_1 then plays another role and we arrive at the well-known case [15] in which the t_j are constant.

Thus, there remains the case in which all three columns are nonzero or one column, for example γ_3, is equal to zero. Suppose $\gamma_3 = 0$; from Eqs. (24) we then have

$$(A^{(1)} - Ea_1)\gamma_1 = 0, \quad (A^{(2)} - Ea_2)\gamma_2 = 0, \quad 0 = 0, \tag{25}$$

$$\begin{cases}\sqrt{2}\,(A^{(4)} - Ea_4)\gamma_2 = 0, \\ (A^{(3)} - Ea_3)\gamma_2 = 0,\end{cases} \qquad \begin{cases}\sqrt{2}\,(A^{(5)} - Ea_5)\gamma_1 = 0, \\ (A^{(3)} - Ea_3)\gamma_1 = 0,\end{cases}$$

$$\begin{cases}\sqrt{2}\,(A^{(6)} - Ea_6)\gamma_1 + (A^{(1)} - Ea_1)\gamma_2 = 0, \\ (A^{(2)} - Ea_2)\gamma_1 + \sqrt{2}\,(A^{(6)} - Ea_6)\gamma_2 = 0,\end{cases}$$

$$\sqrt{2}\,[(A^{(4)} - Ea_4)\gamma_1 + (A^{(5)} - Ea_5)\gamma_2] = 0.$$

It is evident from relations (25) that matrices $A^{(1)}$, $A^{(3)}$, $A^{(5)}$ have a common characteristic vector γ_1, and that matrices $A^{(2)}$, $A^{(3)}$, $A^{(4)}$ have the common characteristic vector γ_2, and that, in addition, the three last equations of (25) are satisfied. In both instances it is first necessary to find the characteristic numbers and vectors of the matrices $A^{(1)}$, $A^{(2)}$, $A^{(3)}$ and then to require that the remaining equations in relations (24) or (25) be satisfied, from which the values of a_4, a_5, a_6 are determined, along with conditions on the elastic moduli so that all of the Eqs. (24) can be satisfied.

A system of the form (23) appears in [18] in a different notation and in which some particular solutions are given.

For a transversally isotropic material with matrix of elastic moduli [19]

$$A_{ij} = \begin{bmatrix} A_{11} & & & & & \\ A_{21} & A_{11} & & & & \\ A_{31} & A_{31} & A_{33} & & \text{sym} & \\ 0 & 0 & 0 & A_{44} & & \\ 0 & 0 & 0 & 0 & A_{44} & \\ 0 & 0 & 0 & 0 & 0 & A_{11} - A_{21} \end{bmatrix}$$

system (1) has the form

$$\left[A_{11}\partial_{11} + \frac{1}{2}(A_{11} - A_{21})\partial_{22} + \frac{1}{2}A_{44}\partial_{33} - \rho\partial_{..}\right]u_1 + \frac{1}{2}(A_{11} + A_{21})\partial_{12}u_2 + \left(\frac{1}{2}A_{44} + A_{31}\right)\partial_{13}u_3 = 0, \tag{26}$$

$$\frac{1}{2}(A_{11} + A_{21})\partial_{21}u_1 + \left[\frac{1}{2}(A_{11} - A_{21})\partial_{11} + A_{11}\partial_{22} + \frac{1}{2}A_{44}\partial_{33} - \rho\partial_{..}\right]u_2 + \left(\frac{1}{2}A_{44} + A_{31}\right)\partial_{23}u_3 = 0,$$

$$\left(\frac{1}{2}A_{44} + A_{31}\right)(\partial_{31}u_1 + \partial_{32}u_2) + \left[\frac{1}{2}A_{44}(\partial_{11} + \partial_{22}) + A_{33}\partial_{33} - \rho\partial_{..}\right]u_3 = 0.$$

By means of the transformation (6), Eqs. (26) are reduced to the equivalent diagonal system (5) in two cases:

$$A_{31} = -\frac{1}{2}A_{44}; \tag{27a}$$

$$\frac{1}{2}A_{44} = \frac{A_{11}A_{33} - A_{31}^2}{A_{11} + A_{33} + 2A_{31}} > 0. \tag{27b}$$

For relation (27a) we then have

$$T = \begin{bmatrix} \partial_1 & -\partial_2 & 0 \\ \partial_2 & \partial_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{28}$$

$$D_1 = A_{11}(\partial_{11} + \partial_{22}) + \frac{1}{2} A_{44}\partial_{33} - \rho\partial_{..},$$
$$D_2 = \frac{1}{2}(A_{11} - A_{21})(\partial_{11} + \partial_{22}) + \frac{1}{2} A_{44}\partial_{33} - \rho\partial_{..},$$
$$D_3 = \frac{1}{2} A_{44}(\partial_{11} + \partial_{22}) + A_{33}\partial_{33} - \rho\partial_{..},$$

and for relation (27b) we have

$$T = \begin{bmatrix} \partial_1 & -\partial_2 & -\alpha\partial_{13} \\ \partial_2 & \partial_1 & -\alpha\partial_{23} \\ \alpha\partial_3 & 0 & \partial_{11} + \partial_{22} \end{bmatrix}, \qquad \alpha = \frac{\frac{1}{2} A_{44} + A_{31}}{A_{11} - \frac{1}{2} A_{44}} = \frac{A_{33} - \frac{1}{2} A_{44}}{\frac{1}{2} A_{44} + A_{31}}; \tag{29}$$

$$D_1 = A_{11}(\partial_{11} + \partial_{22}) + A_{33}\partial_{33} - \rho\partial_{..},$$
$$D_2 = \frac{1}{2}(A_{11} - A_{21})(\partial_{11} + \partial_{22}) + \frac{1}{2} A_{44}\partial_{33} - \rho\partial_{..},$$
$$D_3 = \frac{1}{2} A_{44}\partial_{kk} - \rho\partial_{..}.$$

Operator D_2 and vector $t_{j2} = (-\partial_2, \partial_1, 0)$ are characteristic for the system (26) for all transversally isotropic materials, and not only when the coefficients are connected by conditions (27). For matrices (28) and (29) we obtain, respectively,

$$T'T = TT' = \begin{bmatrix} \partial_{11} + \partial_{22} & & \text{sym} \\ 0 & \partial_{11} + \partial_{22} & \\ 0 & 0 & 1 \end{bmatrix}, \qquad |T| = \partial_{11} + \partial_{22};$$

$$T'T = \begin{bmatrix} \partial_{11} + \partial_{22} + \alpha^2\partial_{33} & & \text{sym} \\ 0 & \partial_{11} + \partial_{22} & \\ 0 & 0 & (\partial_{11} + \partial_{22})(\partial_{11} + \partial_{22} + \alpha^2\partial_{33}) \end{bmatrix},$$

$$TT' = \begin{bmatrix} \partial_{11} + \partial_{22} + \alpha^2\partial_{1133} & & \text{sym} \\ \alpha^2\partial_{1233} & \partial_{11} + \partial_{22} + \alpha^2\partial_{2233} & \\ \alpha\partial_{13}[1 - (\partial_{11} + \partial_{22})] & \alpha\partial_{23}[1 - (\partial_{11} + \partial_{22})] & (\partial_{11} + \partial_{22})^2 + \alpha^2\partial_{33} \end{bmatrix},$$

$$|T| = (\partial_{11} + \partial_{22})(\partial_{11} + \partial_{22} + \alpha^2\partial_{33}).$$

It is evident that for relation (27b) the matrices T and T' are not commutative.

In the examples given the matrices γ_{js} (solutions of Eqs. (23)) for an isotropic material have the form

$$\gamma_{js}^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \gamma_{js}^{(2)} = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix},$$

and for a transversally isotropic material, corresponding to relations (27a) and (27b),

$$\gamma_{js}^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \gamma_{js}^{(2)} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$\gamma_{js}^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \qquad \gamma_{js}^{(2)} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We now find the general structure of the transforming matrix T. Let us assume that there are two characteristic vectors of the form

$$t_{j1} = \alpha_{js}\partial_s, \quad t_{j2} = \beta_{jp}\partial_p. \tag{30}$$

We require them to be orthogonal:

$$t_{j1}t_{j2} = \alpha_{js}\beta_{jp}\partial_{sp} = \frac{1}{2}(\alpha_{js}\beta_{jp} + \alpha_{jp}\beta_{js})\,\partial_{sp} = 0.$$

From this we obtain

$$\frac{1}{2}(\alpha_{js}\beta_{jp} + \alpha_{jp}\beta_{js}) = 0, \tag{31}$$

or, in subscriptless notation, $\alpha'\beta + (\alpha'\beta)' = 0$. It follows from Eq. (31) that $\alpha'\beta = c$ is an antisymmetric matrix ($c' = -c$). If $|\alpha| \neq 0$, then $\beta = (\alpha')^{-1}c$. When relations (31) are satisfied, vectors (30) are orthogonal. The third characteristic vector t_{j3} must be orthogonal to the first two and we can take it in the form $t_{j3} = \varepsilon_{jmn}t_{m1}t_{n2}$. But this formula specifies a vector product; vector $t_3 = t_1 \times t_2$ by definition is orthogonal to t_1 and t_2, and the three vectors form a right-handed triple of vectors. Thus the matrix T of characteristic vectors has the form

$$T = \begin{bmatrix} \alpha_{1s}\partial_s & \beta_{1p}\partial_p & (\alpha_{2s}\beta_{3p} - \alpha_{3s}\beta_{2p})\,\partial_{sp} \\ \alpha_{2s}\partial_s & \beta_{2p}\partial_p & (\alpha_{3s}\beta_{1p} - \alpha_{1s}\beta_{3p})\,\partial_{sp} \\ \alpha_{3s}\partial_s & \beta_{3p}\partial_p & (\alpha_{1s}\beta_{2p} - \alpha_{2s}\beta_{1p})\,\partial_{sp} \end{bmatrix},$$

where the coefficients α_{js}, β_{jp} must satisfy Eqs. (23) and (31). The determinant of matrix T is

$$|T| = t_{i3}t_{i3} = (t_{j1}t_{j1})\,(t_{k2}t_{k2}) = (\alpha_{js}\alpha_{jp}\partial_{sp})\,(\beta_{km}\beta_{kn}\partial_{mn}). \tag{32}$$

Besides solutions of the system (23), (31) for concrete materials, i.e., for given $A_{i(k\ell)j}$, we can present yet another approach in which the $A_{i(k\ell)j}$ are determined upon specifying the characteristic operators $D_{pq} = \tilde{A}_{p(k\ell)q}\partial_{k\ell} - \rho\delta_{pq}\partial_{..}$ ($p = q$) and the vectors (30).

We multiply Eq. (7) by T_{jq}, the algebraic complements of the elements t_{jq}:

$$L_{is}t_{sq}T_{jq} = t_{ip}D_{pq}T_{jq}. \tag{33}$$

Since $t_{sq}T_{jq} = |T|\delta_{sj}$, then from Eq. (33) we obtain

$$L_{ij}\,|T| = t_{ip}D_{pq}T_{jq}. \tag{34}$$

For numerical matrices, when $|T| \neq 0$, from relation (34) we would have

$$L_{ij} = t_{ip}D_{pq}\frac{T_{jq}}{|T|}, \quad \text{where} \quad \frac{T_{jq}}{|T|} = t_{qj}^{-1} = t_{jq}.$$

Since our matrices are operator matrices, we then need to extract the factor $|T|$ on the right side of Eq. (34) or equate all coefficients, on both sides, of the differentiation symbols $\partial_{k\ell pqrs}$. We write Eq. (34) in more detail:

$$(A_{i(kl)j}\partial_{kl} - \rho\delta_{ij}\partial_{..})\,|T| = t_{ip}\,(\tilde{A}_{p(kl)q}\partial_{kl} - \rho\delta_{pq}\partial_{..})\,T_{jq} = t_{ip}\tilde{A}_{p(kl)q}\partial_{kl}T_{jq} - \rho|T|\delta_{ij}\partial_{..}\,.$$

From this we obtain

$$A_{i(kl)j}|T|\partial_{kl} = t_{ip}\tilde{A}_{p(kl)q}T_{jq}\partial_{kl}, \quad p = q. \tag{35}$$

Determining the algebraic complements, we find

$$T_{jq} = [t_{j1}\,(t_{i2}t_{i2}),\ t_{j2}\,(t_{i1}t_{i1}),\ t_{j3}]$$

and we substitute them into relation (35):

$$A_{i(kl)j}|T|\partial_{kl} = [\tilde{A}_{1(kl)1}t_{i1}t_{j1}\,(t_{s2}t_{s2}) + \tilde{A}_{2(kl)2}t_{i2}t_{j2}\,(t_{s1}t_{s1}) + \tilde{A}_{3(kl)3}t_{i3}t_{j3}]\,\partial_{kl}. \tag{36}$$

If Eq. (36) is satisfied, then t_{j1}, t_{j2}, t_{j3} will be characteristic vectors, and $\tilde{A}_{1(k\ell)1}\partial_{k\ell}$, $\tilde{A}_{2(k\ell)2}\partial_{k\ell}$, $\tilde{A}_{3(k\ell)3}\partial_{k\ell}$ are characteristic operators. Actually, from relation (36) we have

$$A_{i(kl)j}t_{j1}|T|\partial_{kl} = [\tilde{A}_{1(kl)1}t_{i1}\,(t_{j1}t_{j1})\,(t_{s2}t_{s2}) + \tilde{A}_{2(kl)2}t_{i2}\,(t_{j2}t_{j1})\,(t_{s1}t_{s1}) + \tilde{A}_{3(kl)3}t_{i3}\,(t_{j3}t_{j1})]\,\partial_{kl} = \tilde{A}_{1(kl)1}t_{i1}|T|\partial_{kl}.$$

There are analogous relations for t_{j2}, t_{j3}. We now substitute expressions (30) and (32) into relation (36):

$$A_{i(kl)j}\alpha_{mp}\alpha_{mq}\beta_{nr}\beta_{ns}\partial_{klpqrs} = \{\tilde{A}_{1(kl)1}\alpha_{ip}\alpha_{jq}\beta_{nr}\beta_{ns} + \tilde{A}_{2(kl)2}\alpha_{mp}\alpha_{mq}\beta_{ir}\beta_{js} + \tilde{A}_{3(kl)3}\varepsilon_{imn}\varepsilon_{jfg}\alpha_{mp}\alpha_{fq}\beta_{nr}\beta_{gs}\}\,\partial_{klpqrs}.$$

In Eq. (37) we now need to equate coefficients of $\partial_{k\ell pqrs}$ with reduction of similar terms taken into account. If we specify the quantities α_{js}, β_{jp}, $\tilde{A}_{p(k\ell)q}$, $p = q$, then from Eq. (37) we can determine the coefficients $A_{i(k\ell)j}$, and, in terms of the latter, the elastic moduli $A_{ik\ell j}$ [14, 15] of all anisotropic materials which admit reduction of system (1) to diagonal form.

For brevity, we write Eq. (37) in the form

$$a_{klpqrs}\partial_{klpqrs} = a_{(klpqrs)}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3} = 0,$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 6$ and $a_{(k\ell pqrs)}$ means that all permutations of the indices in the parentheses are to be carried out and summation taken over the corresponding coefficients; for example $a_{(111112)} = a_{111112} + a_{111121} + a_{111211} + a_{112111} + a_{121111} + a_{211111}$. Relation (37) is then reduced to the equations $a_{(k\ell pqrs)}$. Using a lexicographic arrangement, we write out the possible expressions of $\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ for which equating of coefficients is to be made:

$$\partial_1^6;\ \partial_1^5\partial_2,\ \partial_1^5\partial_3;\ \partial_1^4\partial_2^2,\ \partial_1^4\partial_2\partial_3,\ \partial_1^4\partial_3^2;\ \partial_1^3\partial_2^3,$$
$$\partial_1^3\partial_2^2\partial_3,\ \partial_1^3\partial_2\partial_3^2,\ \partial_1^3\partial_3^3;\ \partial_1^2\partial_2^4,\ \partial_1^2\partial_2^3\partial_3,\ \partial_1^2\partial_2^2\partial_3^2,$$
$$\partial_1^2\partial_2\partial_3^3,\ \partial_1^2\partial_3^4;\ \partial_1\partial_2^5,\ \partial_1\partial_2^4\partial_3,\ \partial_1\partial_2^3\partial_3^2,\ \partial_1\partial_2^2\partial_3^3,$$
$$\partial_1\partial_2\partial_3^4,\ \partial_1\partial_3^5;\ \partial_2^6;\ \partial_2^5\partial_3;\ \partial_2^4\partial_3^2;\ \partial_2^3\partial_3^3;\ \partial_2^2\partial_3^4;\ \partial_2\partial_3^5;\ \partial_3^6.$$

It is evident from this that Eq. (37) is equivalent to 28 independent equations of the form $a_{(k\ell pqrs)} = 0$.

We consider relation (37) when all the indices are identical, i.e., $a_{(111111)} = 0$, $a_{(222222)} = 0$, $a_{(333333)} = 0$:

$$\begin{aligned}A_{i(11)j}\alpha_{m1}\alpha_{m1}\beta_{n1}\beta_{n1} &= \tilde{A}_{1(11)1}\alpha_{i1}\alpha_{j1}\beta_{n1}\beta_{n1} + \\ &+ \tilde{A}_{2(11)2}\alpha_{m1}\alpha_{m1}\beta_{i1}\beta_{j1} + \tilde{A}_{3(11)3}\varepsilon_{imn}\varepsilon_{jfg}\alpha_{m1}\alpha_{f1}\beta_{n1}\beta_{g1},\\ A_{i(22)j}\alpha_{m2}\alpha_{m2}\beta_{n2}\beta_{n2} &= \tilde{A}_{1(22)1}\alpha_{i2}\alpha_{j2}\beta_{n2}\beta_{n2} + \\ &+ \tilde{A}_{2(22)2}\alpha_{m2}\alpha_{m2}\beta_{i2}\beta_{j2} + \tilde{A}_{3(22)3}\varepsilon_{imn}\varepsilon_{jfg}\alpha_{m2}\alpha_{f2}\beta_{n2}\beta_{g2},\\ A_{i(33)j}\alpha_{m3}\alpha_{m3}\beta_{n3}\beta_{n3} &= \tilde{A}_{1(33)1}\alpha_{i3}\alpha_{j3}\beta_{n3}\beta_{n3} + \\ &+ \tilde{A}_{2(33)2}\alpha_{m3}\alpha_{m3}\beta_{i3}\beta_{j3} + \tilde{A}_{3(33)3}\varepsilon_{imn}\varepsilon_{jfg}\alpha_{m3}\alpha_{f3}\beta_{n3}\beta_{g3}.\end{aligned} \quad (38)$$

If $\alpha_{m1}\alpha_{m1} \neq 0$, $\beta_{n1}\beta_{n1} \neq 0$, then from the first of Eqs. (38) we obtain

$$A_{i(11)j} = \tilde{A}_{1(11)1}\frac{\alpha_{i1}\alpha_{j1}}{\alpha_{m1}\alpha_{m1}} + \tilde{A}_{2(11)2}\frac{\beta_{i1}\beta_{j1}}{\beta_{n1}\beta_{n1}} + \tilde{A}_{3(11)3}\frac{\varepsilon_{imn}\alpha_{m1}\beta_{n1}\varepsilon_{jfg}\alpha_{f1}\beta_{g1}}{\alpha_{m1}\alpha_{m1}\beta_{n1}\beta_{n1}}. \quad (39)$$

If $\alpha_{m2}\alpha_{m2} \neq 0$, $\beta_{n2}\beta_{n2} \neq 0$, and $\alpha_{m3}\alpha_{m3} \neq 0$, $\beta_{n3}\beta_{n3} \neq 0$, then from Eqs. (38) we obtain

$$\begin{aligned}A_{i(22)j} &= \tilde{A}_{1(22)1}\frac{\alpha_{i2}\alpha_{j2}}{\alpha_{m2}\alpha_{m2}} + \tilde{A}_{2(22)2}\frac{\beta_{i2}\beta_{j2}}{\beta_{n2}\beta_{n2}} + \tilde{A}_{3(22)3}\frac{\varepsilon_{imn}\alpha_{m2}\beta_{n2}\varepsilon_{jfg}\alpha_{f2}\beta_{g2}}{\alpha_{m2}\alpha_{m2}\beta_{n2}\beta_{n2}},\\ A_{i(33)j} &= \tilde{A}_{1(33)1}\frac{\alpha_{i3}\alpha_{j3}}{\alpha_{m3}\alpha_{m3}} + \tilde{A}_{2(33)2}\frac{\beta_{i3}\beta_{j3}}{\beta_{n3}\beta_{n3}} + \tilde{A}_{3(33)3}\frac{\varepsilon_{imn}\alpha_{m3}\beta_{n3}\varepsilon_{jfg}\alpha_{f3}\beta_{g3}}{\alpha_{m3}\alpha_{m3}\beta_{n3}\beta_{n3}}.\end{aligned} \quad (40)$$

But formulas (39) and (40) are representations of the matrices $A_{i(11)j}$, $A_{i(22)j}$, $A_{i(33)j}$ in terms of the characteristic numbers and vectors. Taking note of Eq. (31), it is not hard to verify that $\tilde{A}_{1(11)1}$, $\tilde{A}_{2(11)2}$, $\tilde{A}_{3(11)3}$ are characteristic numbers and that α_{j1}, β_{j1}, $\varepsilon_{jfg}\alpha_{f1}\beta_{g1}$

are characteristic vectors of the matrix $A_{i(11)j}$ [see Eq. (39)]. Similar statements apply for $A_{i(22)j}$, $A_{i(33)j}$ [see Eq. (40)].

Since the symmetry conditions $A_{i(k\ell)j} = A_{k(ij)\ell}$ [14, 15] hold, they then impose additional constraints on the quantities on the right sides of Eqs. (39) and (40):

$$\begin{gathered}
\bar{A}_{1(11)1} \frac{\alpha_{21}^2}{\alpha_{m1}\alpha_{m1}} + \bar{A}_{2(11)2} \frac{\beta_{21}^2}{\beta_{n1}\beta_{n1}} + \bar{A}_{3(11)3} \frac{(\alpha_{31}\beta_{11} - \alpha_{11}\beta_{31})^2}{\alpha_{m1}\alpha_{m1}\beta_{n1}\beta_{n1}} = \\
= \bar{A}_{1(22)1} \frac{\alpha_{12}^2}{\alpha_{m2}\alpha_{m2}} + \bar{A}_{2(22)2} \frac{\beta_{12}^2}{\beta_{n2}\beta_{n2}} + \bar{A}_{3(22)3} \frac{(\alpha_{22}\beta_{32} - \alpha_{32}\beta_{22})^2}{\alpha_{m2}\alpha_{m2}\beta_{n2}\beta_{n2}}, \\
\bar{A}_{1(11)1} \frac{\alpha_{31}^2}{\alpha_{m1}\alpha_{m1}} + \bar{A}_{2(11)2} \frac{\beta_{31}^2}{\beta_{n1}\beta_{n1}} + \bar{A}_{3(11)3} \frac{(\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11})^2}{\alpha_{m1}\alpha_{m1}\beta_{n1}\beta_{n1}} = \\
= \bar{A}_{1(33)1} \frac{\alpha_{13}^2}{\alpha_{m3}\alpha_{m3}} + \bar{A}_{2(33)2} \frac{\beta_{13}^2}{\beta_{n3}\beta_{n3}} + \bar{A}_{3(33)3} \frac{(\alpha_{23}\beta_{33} - \alpha_{33}\beta_{23})^2}{\alpha_{m3}\alpha_{m3}\beta_{n3}\beta_{n3}}, \\
\bar{A}_{1(22)1} \frac{\alpha_{32}^2}{\alpha_{m2}\alpha_{m2}} + \bar{A}_{2(22)2} \frac{\beta_{32}^2}{\beta_{n2}\beta_{n2}} + \bar{A}_{3(22)3} \frac{(\alpha_{12}\beta_{22} - \alpha_{22}\beta_{12})^2}{\alpha_{m2}\alpha_{m2}\beta_{n2}\beta_{n2}} = \\
= \bar{A}_{1(33)1} \frac{\alpha_{23}^2}{\alpha_{m3}\alpha_{m3}} + \bar{A}_{2(33)2} \frac{\beta_{23}^2}{\beta_{n3}\beta_{n3}} + \bar{A}_{3(33)3} \frac{(\alpha_{33}\beta_{13} - \alpha_{13}\beta_{33})^2}{\alpha_{m3}\alpha_{m3}\beta_{n3}\beta_{n3}}.
\end{gathered} \tag{41}$$

If we specify matrices in accordance with formulas (39) and (40), the quantities on the right sides must then be subject to conditions (31) and (41).

It is evident from Eqs. (39) and (40) that the corresponding matrices of characteristic vectors for $A_{i(11)j}$, $A_{i(22)j}$, $A_{i(33)j}$, under the direct approach, need to be written (to be numbered) thus:

$$[\alpha_{j1}, \beta_{j1}, \varepsilon_{jfg}\alpha_{f1}\beta_{g1}], \quad [\alpha_{j2}, \beta_{j2}, \varepsilon_{jfg}\alpha_{f2}\beta_{g2}], \quad [\alpha_{j3}, \beta_{j3}, \varepsilon_{jfg}\alpha_{f3}\beta_{g3}], \tag{42}$$

in order for conditions (31) and (41) to be satisfied. Further, the matrices α_{js}, β_{jp} are constructed from the first two columns of matrices (42) and from them possible characteristic vectors for the operators L_{ij} are obtained:

$$t_{j1} = \alpha_{js}\partial_s, \quad t_{j2} = \beta_{jp}\partial_p, \quad t_{j3} = \varepsilon_{jmn}t_{m1}t_{n2}. \tag{43}$$

The characteristic operators must have the form

$$\begin{aligned}
D_{11} &= \bar{A}_{1(kl)1}\partial_{kl} - \rho\partial_{..} = (\bar{A}_{1(11)1}\partial_{11} + \bar{A}_{1(22)1}\partial_{22} + \bar{A}_{1(33)1}\partial_{33}) + \\
&\quad + 2\bar{A}_{1(23)1}\partial_{23} + 2\bar{A}_{1(13)1}\partial_{13} + 2\bar{A}_{1(12)1}\partial_{12} - \rho\partial_{..}, \\
D_{22} &= \bar{A}_{2(kl)2}\partial_{kl} - \rho\partial_{..} = (\bar{A}_{2(11)2}\partial_{11} + \bar{A}_{2(22)2}\partial_{22} + \bar{A}_{2(33)2}\partial_{33}) + \\
&\quad + 2\bar{A}_{2(23)2}\partial_{23} + 2\bar{A}_{2(13)2}\partial_{13} + 2\bar{A}_{2(12)2}\partial_{12} - \rho\partial_{..}, \\
D_{33} &= \bar{A}_{3(kl)3}\partial_{kl} - \rho\partial_{..} = (\bar{A}_{3(11)3}\partial_{11} + \bar{A}_{3(22)3}\partial_{22} + \bar{A}_{3(33)3}\partial_{33}) + \\
&\quad + 2\bar{A}_{3(23)3}\partial_{23} + 2\bar{A}_{3(13)3}\partial_{13} + 2\bar{A}_{3(12)3}\partial_{12} - \rho\partial_{..}.
\end{aligned} \tag{44}$$

Thus, knowing the characteristic numbers and vectors of matrices $A_{i(11)j}$, $A_{i(22)j}$, $A_{i(33)j}$, we can determine the characteristic vectors (43) of the operators L_{ij} and the partial characteristic operators (terms in parentheses in Eqs. (44)). We obtain the remaining coefficients in Eqs. (44) from the condition that expression (43) be characteristic vectors, acting directly on L_{ij} or requiring satisfaction of the remaining equations of system (23) or (37).

As can be verified, expressions (39), (40) satisfy the first three of Eqs. (23). The fourth and fifth of Eqs. (23) will be satisfied if we take the block matrix in the form

$$\begin{bmatrix} 2A_{i(23)j}, & A_{i(22)j} \\ A_{i(33)j}, & 2A_{i(23)j} \end{bmatrix} = \begin{bmatrix} 2\bar{A}_{1(23)1}\alpha_{i2} + \bar{A}_{1(22)1}\alpha_{i3} \\ \bar{A}_{1(33)1}\alpha_{i2} + 2\bar{A}_{1(23)1}\alpha_{i3} \end{bmatrix} \frac{[\alpha_{j2}, \alpha_{j3}]}{\alpha_{k2}\alpha_{k2} + \alpha_{k3}\alpha_{k3}} + \tag{45}$$

$$+\begin{bmatrix}2\bar{A}_{2(23)2}\beta_{i2}+\bar{A}_{2(22)2}\beta_{i3}\\ \bar{A}_{2(33)2}\beta_{i2}+2\bar{A}_{2(23)2}\beta_{i3}\end{bmatrix}\frac{[\beta_{j2},\ \beta_{j3}]}{\beta_{k2}\beta_{k2}+\beta_{k3}\beta_{k3}}+\begin{bmatrix}2\bar{A}_{3(23)3}\varepsilon_{imn}\alpha_{m2}\beta_{n2}+\bar{A}_{3(22)3}\varepsilon_{imn}\alpha_{m3}\beta_{n3}\\ \bar{A}_{3(33)3}\varepsilon_{imn}\alpha_{m2}\beta_{n2}+2\bar{A}_{3(23)3}\varepsilon_{imn}\alpha_{m3}\beta_{n3}\end{bmatrix}\frac{[\varepsilon_{jmn}\alpha_{m2}\beta_{n2},\ \varepsilon_{jmn}\alpha_{m3}\beta_{n3}]}{\alpha_{k2}\alpha_{k2}\beta_{q2}\beta_{q2}+\alpha_{k3}\alpha_{k3}\beta_{q3}\beta_{q3}}.$$

Similar solutions may be written also for the remaining Eqs. of (23). Expressions of the corresponding matrices $A_{i(k\ell)j}$, obtained from expressions (39), (40), (45) and analogous solutions of Eqs. (23), must coincide among themselves. Moreover, it is also necessary that the symmetry conditions $A_{i(k\ell)j} = A_{k(ij)\ell}$ [14, 15] be satisfied. All of this imposes additional constraints (of the type (41)) on the quantities on the right side of Eq. (45). Owing to their complexity, we shall not list them here.

Thus, the approach presented here allows us, in principle, to determine all anisotropic materials permitting reduction of system (1) to diagonal form. The formulation of boundary value problems for the functions v_j is the object of special investigations.

If under the transformations of system (1) we allow operators with variable coefficients, then it is necessary to use, instead of transposed matrices, conjugate matrices of operators [20]. For operators of the form

$$A_{ij} = a_{ij}(x_s) + a_{ijk}(x_s)\,\partial_k + a_{ij(kl)}(x_s)\,\partial_{kl} + a_{ij(klm)}(x_s)\,\partial_{klm} + \dots$$

the formally conjugate operator

$$A^*_{ji} = a_{ij} - \partial_k a_{ijk} + \partial_{kl} a_{ij(kl)} - \partial_{klm} a_{ij(klm)} + \dots$$

Let $A^* = A$, $D^* = D$ and $AC = BD$; then $C^*A = DB^*$. If $u = C\varphi$, where $D\varphi = 0$, the equation $Au = AC\varphi = BD\varphi = 0$ is then satisfied. But if $\varphi = B^*\tilde{u}$, where $A\tilde{u} = 0$ the equation $D\varphi = DB^*\tilde{u} = C^*A\tilde{u} = 0$ is then satisfied.

If $A\tilde{u} = 0$, then $u = CB^*\tilde{u}$ is also a solution $Au = ACB^*\tilde{u} = BDB^*\tilde{u} = BC^*A\tilde{u} = 0$.

For an isotropic material, in the case of statics, the operators have the form

$$A_{ij} = \partial_{ij} + \mu_1\delta_{ij}\partial_{ss} = A^*_{ji},\quad \mu_1 = \frac{\mu}{\lambda+\mu},$$
$$C_{jk} = (1+2\mu_1)\,\delta_{jk} - x_k\partial_j,\quad C^*_{kj} = 2(1+\mu_1)\,\delta_{jk} + x_k\partial_j,$$
$$B_{ij} = (2\mu_1 - 1)\,\delta_{ij} - x_j\partial_i,\quad B^*_{ji} = 2\mu_1\delta_{ij} + x_j\partial_i,$$
$$D_{jk} = (1+\mu_1)\,\delta_{jk}\partial_{pp} = D^*_{kj},\quad x_4 = 1,\quad \partial_4 = 0,$$

where the relations $AC = BD$, $C^*A = DB^*$ are satisfied. We now write the well-known Papkovich–Neiber solution [1] as follows:

$$u_j = C_{jk}\varphi_k = (1+2\mu_1)\,\varphi_j - x_1\partial_j\varphi_1 - x_2\partial_j\varphi_2 - x_3\partial_j\varphi_3 - \partial_j\varphi_4,$$
$$D_{jk}\varphi_k = (1+\mu_1)\,\partial_{pp}\varphi_j = 0. \tag{46}$$

Expressions of functions φ_j in terms of a solution of the Lamé equations is as follows:

$$\varphi_j = B^*_{ji}\tilde{u}_i = 2\mu_1\tilde{u}_j + x_j\partial_i\tilde{u}_i,\quad \tilde{u}_4 = 0,$$
$$A_{ij}\tilde{u}_j = \partial_{ij}\tilde{u}_j + \mu_1\partial_{ss}\tilde{u}_i = 0. \tag{47}$$

Again, let us write out a formula for the production of new solutions:

$$u_j = C_{jk}B^*_{ki}\tilde{u}_i = \{2\mu_1[(1+2\mu_1)\,\delta_{ji} + x_j\partial_i - x_i\partial_j] - x_kx_k\partial_{ji}\}\,\tilde{u}_i =$$
$$= 2\mu_1[(1+2\mu_1)\,\tilde{u}_j + x_j\partial_i\tilde{u}_i - x_i\partial_j\tilde{u}_i] - (x_1^2 + x_2^2 + x_3^2 + 1)\,\partial_{ji}\tilde{u}_i.$$

Here $A\tilde{u} = 0$.

Formulas (46) and (47) solve the long-discussed problem concerning completeness and generality of the Papkovich–Neiber solution. It follows from relations (47) that

$$\varphi_j = 2\mu_1\tilde{u}_j + x_j\varphi_4,\quad j = 1, 2, 3,\quad \varphi_4 = \partial_i\tilde{u}_i,$$
$$\partial_i\varphi_j = 2\mu_1\partial_i\tilde{u}_j + \delta_{ji}\varphi_4 + x_j\partial_i\varphi_4;$$
$$\partial_i\varphi_i = (2\mu_1 + 3)\,\varphi_4 + x_i\partial_i\varphi_4, \tag{48}$$

i.e., functions φ_j are interrelated through the relation (48).

Remark. The formulas presented here do not exhaust all solutions of system (1). For complete generality it is necessary to consider the equations Dv = f, Tf = 0 or Dφ = f, Bf = 0. The Papkovich–Neiber solution (46) is a general solution since, as a direct verification shows, it satisfies the condition of generality D Ker C = Ker B [20].

LITERATURE CITED

1. P. F. Papkovich, Theory of Elasticity [in Russian], Oborongiz, Leningrad–Moscow (1939).
2. B. G. Galerkin, Collected Works [in Russian], Vol. 1, Akad. Nauk SSSR, Moscow (1952).
3. G. Neiber, Stress Concentration [in Russian], OGIZ, Gostekhizdat, Moscow–Leningrad (1947).
4. Yu. A. Krutkov, Stress Function Tensor and General Solutions in Statics in Elasticity Theory [in Russian], Akad. Nauk SSSR, Moscow–Leningrad (1949).
5. N. I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity, Noordhoff, Groningen (1953).
6. E. N. Baida, General Solutions of the Theory of Elasticity and Problems Concerning a Parallelepiped and a Cylinder [in Russian], Gosstroiizdat, Leningrad (1961).
7. S. G. Lekhnitskii, Theory of Elasticity of an Anisotropic Elastic Body, Holden-Day, San Francisco (1963).
8. N. I. Ostrosablin, "General representation of a solution of the equations of linear elastic theory for an isotropic body," in: Dynamics of a Continuous Medium: Compendium of Scientific Papers, No. 61, Inst. of Hydrodynamics, Siberian Branch [in Russian], Acad. of Sciences of the USSR (1983).
9. N. I. Ostrosablin, "On a general solution of the equations of the linear theory of elasticity," in: Dynamics of a Continuous Medium: Compendium of Scientific Papers, No. 92, Inst. of Hydrodynamics, Siberian Branch [in Russian], Acad. of Sciences of the USSR (1991).
10. K. Marguerre, "Ansätze zur Lösung der Grundgleichungen der Elastizitätstheorie," ZAMM, 35, No. 6/7 (1955).
11. C. Truesdell, "Invariant and complete stress functions for general continua," Arch. Rat. Mech. Anal., 4, No. 1 (1959).
12. N. I. Ostrosablin and S. I. Senashov, "General solutions and symmetries of equations of the linear theory of elasticity," Dokl. Akad. Nauk SSSR, 322, No. 3 (1992).
13. A. N. Norris, "On the acoustic determination of the elastic moduli of anisotropic solids and acoustic conditions for the existence of symmetry planes," Quart. J. Mech. Appl. Math., 42, No. 3 (1989).
14. N. I. Ostrosablin, "On the matrix of coefficients in the equations of the linear theory of elasticity," Dokl. Akad. Nauk SSSR, 321, No. 1 (1991).
15. N. I. Ostrosablin, "On the equations of the linear theory of elasticity," Prikl. Mekh. Tekh. Fiz., No. 3, May-June (1992).
16. V. M. Borok, "On systems of linear partial differential equations with constant coefficients," Izv. Vyssh. Uchebn. Zaved., Mat. No. 1 (1957).
17. L. V. Ovsyannikov, Group Analysis of Differential Equations [in Russian], Nauka, Moscow (1978).
18. J. J. Marciniak, "The generalized scalar wave equation and linear differential invariants in linear elasticity," Intern. J. Engng. Sci., 27, No. 6 (1989).
19. N. I. Ostrosablin, "Characteristic moduli of elasticity and states for materials of crystallographic syngony," in: Dynamics of a Continuous Medium: Compendium of Scientific Papers, No. 75, Inst. of Hydrodynamics, Siberian Branch [in Russian], Academy of Sciences of the USSR (1986).
20. Zhang Hong-qing and Yang Guang, "Constructions of the general solution for a system of partial differential equations with variable coefficients," Appl. Math. and Mech. (Engl. Ed.), 12, No. 2 (1991).